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Unexpected thresholds from Independence of Irrelevant Alternatives in Fuzzy Arrow Theorems

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Summary

It is well known Arrow Theorem and its impact into Social Choice. It states that under an apparently mild set of conditions no rule fusing individual preferences into a social one is possible. In order to solve this situation, a possibility is to skip from dichotomic preferences to fuzzy ones. All conditions imposed to aggregation rules should be adapted to the fuzzy setting and due to the existence of different generalizations for each condition, depending on the chosen combination, a possibility or an impossibility result arises.
In addition, in case we find a reasonable fuzzy aggregation rule, in most situations dichotomic decisions have to be taken at the end of the day, so the use of thresholds over fuzzy preferences is compulsory to make any decision. Surprisingly, independence of irrelevant alternatives axioms induce different thresholds which, besides they can be used on discrete and dichotomic decision making, transform fuzzy spaces of preferences and its aggregation functions into discrete ones allowing the application of new techniques to their study.

Classic representation of preferences

Give a set X including all alternatives involved in a decision (e.g. job candidates, toasters to buy,...). They can be ordered by using binary relations satisfying certain properties. Particularly these binary relations can be preorders.

Definition A preorder $\succsim$ on X is a reflexive $(\forall x \in X\ x \succsim x)$ and transitive $(\forall x, y, z \in X\ [x \succsim y \wedge y \succsim z \Rightarrow x \succsim z])$ binary relation. Additionally, it is said that $\succsim$ is a total preorder if $(\forall x, y \in X\ x \succsim y \vee y \succsim x)$.
To give a total preorder on X is equivalent to give a ranking with ties on X.

$\succsim$	Preorder	$x \succsim y$: x is at least as good as y
		$x \succ y$: x is better than y
$\downarrow$	$\downarrow$	$x \sim y$: x and y are equally preferred
$(\succ, \succsim, \sim)$	Preference	$x \succ y \Leftrightarrow x \succsim y \wedge \neg(y \succsim x)$
		$x \sim y \Leftrightarrow x \succsim y \wedge y \succsim x$

Arrovian model and Theorem of Impossibility

Arrow in [1] proved that given a finite set of agents $N = \{1, \dots, n\}$, each one expressing their preferences over a set of alternatives X with total preorders, there is no *"fair"* rule which aggregates all individual preferences obtaining a social one. Formally, if the set of all total preorders on X is denoted by $\mathcal{O}_X$:

Theorem There is no function $f : \mathcal{O}_X^n \to \mathcal{O}_X$ on a set of alternatives with $|X| \geq 3$ satisfying for every $x, y \in X$ and profiles $(\succsim_j), (\succsim'_j) \in \mathcal{O}_X^n$, the following conditions:

- Strong Paretian: $\forall i \in N\ x \succsim_i y \wedge \exists k \in N\ x \succ_k y \Rightarrow x \succ_{f((\succsim_j))} y$
- Independence of irrelevant alternatives (**IIA**):
$$\left[\forall i \in N\ \ \succsim_{i \rceil \{x,y\}} = \succsim'_{i \rceil \{x,y\}}\right] \Rightarrow f((\succsim_j))_{\rceil \{x,y\}} = f((\succsim'_j))_{\rceil \{x,y\}}$$
- Non dictatoriship: $\nexists k \in N \left[x \succ_k y \Rightarrow x \succ_{f((\succsim_j))} y\right]$

Could Arrovian impossibility be walked around by using fuzzy preferences instead of dichotomic ones?

First of all, fuzzy preferences have to be defined. The triplet $(\succ, \succsim, \sim)$ is generalized to fuzzy relations satisfying certain properties.

$$\underset{\text{Dichotomic Preference}}{(\succ, \succsim, \sim)} \longrightarrow \underset{\text{Fuzzy Preference}}{(P, R, I) =: \Lambda}$$

$$P, R, I : X \times X \to [0, 1]$$

P is asymmetric $[P(x, y) > 0 \Rightarrow P(y, x) = 0]$
I is symmetric $[I(x, y) = I(y, x)]$
$R(x, y) > R(y, x) \Leftrightarrow P(x, y) > 0$
(and more properties...)

Properties of preferences $(\succ, \succsim, \sim)$ can be generalized to the fuzzy setting in different ways. For example:

$$\text{Transitivity} \to \begin{cases} T\text{-transitivity (with } T \text{ a t-norm)} \\ \quad [\forall x, y, z \in X\ R(x, z) \geq T(R(x, y), R(y, z))] \\ \text{Weak transitivity} \\ \quad [\forall x, y, z \in X\ R(x, y) \geq R(y, x) \wedge R(y, z) \geq R(z, y) \Rightarrow R(x, z) \geq R(z, x)] \end{cases}$$

$$\text{Total} \to \begin{cases} \cup\text{-connected (with } \cup \text{ a union or a conorm)} \\ \quad [\forall x, y \in X\ R(x, y) \cup R(y, x) = 1] \end{cases}$$

If a set of fuzzy preferences on X is denoted by $\mathcal{FP}$, an n-aggregation fuzzy rule is a function $f : \mathcal{FP}^n \to \mathcal{FP}$. Arrow axioms can be also generalized in various ways. For example:

$$\text{Strong Paretian} \to \begin{cases} \text{Weak Pareto: } \forall x, y \in X\ P_i(x, y) > 0 \Rightarrow P_f(x, y) > 0 \\ \text{Pareto: } \forall x, y \in X\ P_f(x, y) \geq \min_{i \in N} P_i(x, y) \end{cases}$$

$$\text{Dictatorship} \to \begin{cases} \text{Dictatorship: } \exists k \in N\ P_k(x, y) > 0 \Rightarrow P_f(x, y) > 0 \\ \alpha-\text{dictatorship: } \exists k \in N\ \forall t \in [0, 1] P_k(x, y) > t \Rightarrow P_f(x, y) > t \end{cases}$$

But, how to generalize IIA axiom?

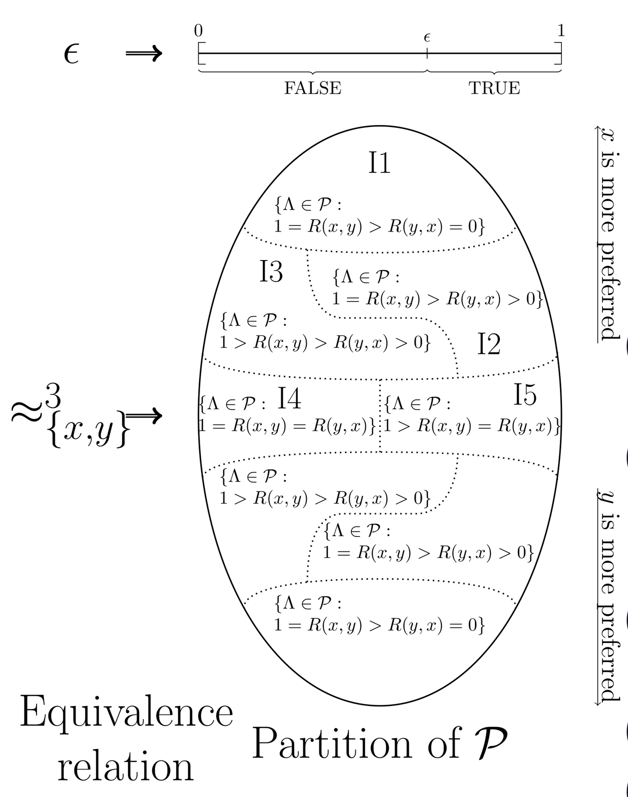
This axiom represents the idea that the social choice between any pair of alternatives only relies on individual preferences over the same pair of alternatives.
There are different extensions to the fuzzy environment in the literature, but all are defined by equivalence relations $\{\approx_{\{x,y\}}\}_{x,y \in X}$ on preferences following the schema below:

$$\forall x, y \in X\ [\forall i \in N \Lambda_i \approx_{\{x,y\}} \Lambda'_i \Rightarrow f((\Lambda_i)) \approx_{\{x,y\}} f((\Lambda'_i))]$$

In other words, if two profiles are the same in some sense (defined by the equivalence relations) with respect to a pair of alternatives, then their aggregations also have to be the same with respect to the same pair. Here are some examples:

- $\Lambda \approx^1_{\{x,y\}} \Lambda' \Leftrightarrow \Lambda_{\rceil \{x,y\}} = \Lambda'_{\rceil \{x,y\}}$
- $\Lambda \approx^2_{\{x,y\}} \Lambda' \Leftrightarrow supp(R_{\rceil \{x,y\}}) = supp(R'_{\rceil \{x,y\}})$
- $\Lambda \approx^3_{\{x,y\}} \Lambda' \Leftrightarrow \Lambda \approx^2_{\{x,y\}} \wedge [\forall \overline{z}, \overline{z}' \in \{x, y\}^2 R(\overline{z}) > R(\overline{z}') \Leftrightarrow R'(\overline{z}) > R(\overline{z}')]$

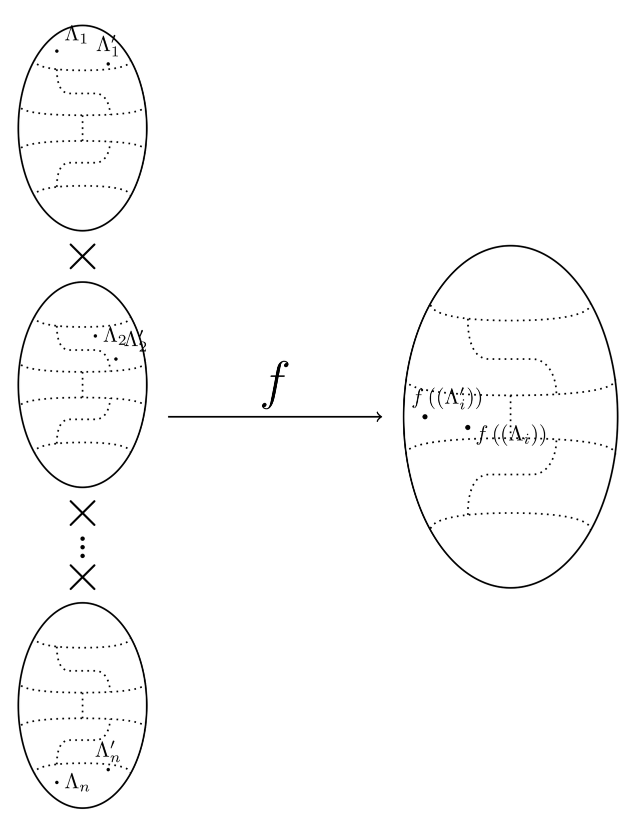
Each equivalence induces a partition on the preference space $\mathcal{FP}$ by means of equivalence classes. These can be read as a qualitative discrimination between preferences belonging to different equivalence classes.



Notice that this qualitative discimination can be more complex than usual thresholds $\{FALSE, TRUE\}$ in the fuzzy scale $[0, 1]$ defined by a boundary $\epsilon \in (0, 1)$. Consider the partition generated by $\approx^3_{\{x,y\}}$ in $\mathcal{P}$, the set of weak transitive and $\cup$-complete fuzzy preferences. They can be interpreted as follows:

(I1) The agent completely prefers x over y because, under their point of view, clearly y is too bad compared to x.
(I2) The agent thinks that x is better than y, but she is not completely sure about that and thinks further analysis could change her preferences.
(I3) The agent thinks that x is better than y but y is not clearly much worse than x.
(I4) The agent is strongly sure that x and y are indifferent.
(I5) It seems to the agent that x and y are equivalent.

If preferences Λ_i and Λ'_i are in the same component for all $i \in N$, IIA axiom guarantees, as it is illustrated in the figure, that $f((\Lambda_i))$ and $f((\Lambda'_i))$ have to be in the same component too.
We can study the action of f over the whole components instead of preferences, individually. This change from a continuous paradigm (infinite fuzzy preferences) to a discrete one (few components) is important because it allows to introduce easily classical social choice and combinatorial techniques, or Arrow's Theorem in the study of discretization of aggregation functions, which at the same time will provide information from the initial fuzzy aggregation functions. For example, the following impossibility theorem can be proved using this approach:



Theorem: Let $f : \mathcal{P}^n \to \mathcal{P}$ be a fuzzy aggregation function satisfying IIA defined by $\{\approx^3_{\{x,y\}}\}_{x,y \in X}$ and weakly paretian, then f is dictatorial.

Future research

- Investigate the partitions induced by other IIA axioms and find methods using discretizations leading to possibility results, not just impossibility ones.
- Propose other kinds of axioms supplying independence of irrelevant alternatives not based on equivalence relations.

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